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THE ESTIMATION OF THE GRAPH BOX DIMENSION OF A CLASS OF FRACTALS

ALINA BĂRBULESCU

"Ovidius" University, Constanta, Romania

Abstract
Fractal dimensions are the most important attributes of fractals and the Box counting dimension is widely used. Usually it is not so easy to determine dimensions. In some papers we have considered a class of functions and we have studied the finitude of the Hausdorff h-measures of the graph, Γ, of an element of this class. In this paper we determine the Box dimension of Γ.

***Keywords**: Box dimension, Haudorff h-measure, Hadamard condition*

1. INTRODUCTION

The importance of the fractal sets in sciences increases in the last years. The most important attributes of fractals are the dimensions.

For the Besicovitch functions, given by

$$B(t) = \sum_{k=1}^{\infty} \lambda_k^{s-2} \cos \lambda_k t , \tag{1}$$

where $1 < s < 2$, $\lambda > 0$ and $\lim\limits_{k \to \infty} \lambda_k = \infty$, the fractal dimension have been estimated, in some cases ([5]), but their exact fractal dimension is unknown.

Definition 1 Let $\mathbf{R}^n$ be the Euclidean n - dimensional space. If $r_0 > 0$ is a given number, then, a continuous function $h(r)$, defined on $[0, r_0)$, nondecreasing and such that $\lim\limits_{r \to 0} h(r) = 0$ is called a measure function.

If $\delta > 0, E$ is a nonempty and bounded subset of $\mathbf{R}^n$ and h is a measure function then, the Hausdorff h - measure of E is defined by:

$$H_h(E) = \lim_{\delta \to 0} \left\{ \inf \sum_i h(\rho_i) \right\},$$

inf being considered over all covers of E with a countable number of spheres of radii $\rho_i \le \delta$.

Particularly, when $h(r) = r^s$, the obtained measure is called the s-dimensional Hausdorff measure and is denoted by H_s.

Definition 2 It is said that the sequence $\{\lambda_i\}_{i\in\mathbf{N}^*}$ satisfies the Hadamard condition if there exists $\varepsilon > 1$ such that $\lambda_{i+1} > \varepsilon\lambda_i$, for every $i \in \mathbf{N}^*$.

It is known that the graph of a function $f : D \to \mathbf{R}$ is the set

$$\Gamma(f) = \{(x, f(x)),\ x \in D\}.$$

In our papers ([1] - [4]), the function $\cos\lambda_k t$ from (1) was replaced

$$g(x) = \begin{cases} 2x & , 0 \le x < \frac{1}{2} \\ -2(x-1), & \frac{1}{2} \le x < \frac{3}{2} \\ 2(x-2) & , \frac{3}{2} \le x < 2 \end{cases} \tag{2}$$

and the following function was introduced

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{s-2} g(\lambda_i x), (\forall) x \in [0,1],\ s \in [1,2), \tag{3}$$

where g is given in (2) and $\{\lambda_i\}_{i\in\mathbf{N}^*} \in \mathbf{R}_+$ is a sequence that satisfies the Hadamard condition.

Theorem 1 ([1], [3]) If h is a measure function such that $h(t) \sim t^p$, $p \ge 2$, f is the function defined in (3), $s \in [1,2)$ and $\{\lambda_i\}_{i\in\mathbf{N}^*} \in \mathbf{R}_+$ is a sequence that satisfies Hadamard condition, then $H(\Gamma(f)) < \infty$. The result remains true if $p \ge 1$ and $\delta > 1$.

In what follows we shall determine the Box dimension of the graph of the function given in (3).

There are many equivalent definitions ([6]) of the Box dimension, but we shall use the following one.

Definition 3 Let β be a positive number and let E be a nonempty and bounded subset of $\mathbf{R}^2$. Consider the β - mesh of $\mathbf{R}^2$,

$$\{[i\beta, (i+1)\beta] \times [j\beta, (j+1)\beta],\ i, j \in \mathbf{N}\}$$

If $N_\beta(E)$ is the number of the β - mesh squares that intersect E, then the upper and lower Box dimension of E are defined by

$$\overline{\dim}_B E = \overline{\lim_{\beta\to 0}} \frac{\log N_\beta(E)}{-\log\beta};\ \underline{\dim}_B E = \underline{\lim_{\beta\to 0}} \frac{\log N_\beta(E)}{-\log\beta}.$$

If these limits are equal, the common value is called the Box dimension of E and is denoted by $\dim_B E$.

For any given function $f : [0,1] \to \mathbf{R}$ and $[t_1, t_2] \subset [0,1]$, we denote by $R_f[t_1, t_2]$ the oscillation of f on the interval $[t_1, t_2]$, that is $R_f[t_1, t_2] = \sup_{t_1 \le t,\ u \le t_2} |f(t) - f(u)|$.

For briefly, any $C, C_1, \ldots, C_5$ in this paper indicates a positive constant that may have different values.

2. RESULTS

In this part of the paper we shall use the following results:

Lemma 1 ([6]). Let $f:[0,1]\to \mathbf{R}$ be a continuous function, $0<\beta<1$ and m be the least integer greater than or equal to β^{-1}. If N_β is the number of the squares of the β - mesh that intersects $\Gamma(f)$, then

$$\beta^{-1}\sum_{j=0}^{m-1} R_f\left[j\beta,(j+1)\beta\right]\le N_\beta \le 2m+\beta^{-1}\sum_{j=0}^{m-1} R_f\left[j\beta,(j+1)\beta\right].$$

Lemma 2 (Hölder inequality). If $k\in \mathbf{N}^*$, $a_i, b_i\in \mathbf{R}$, $i\in\overline{1,k}$ and $0<p<1$, $q=\frac{p}{p-1}$, then

$$\sum_{i=1}^{k}\left|a_i b_i\right|\ge\left(\sum_{i=1}^{k}\left|a_i\right|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}\left|b_i\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2 If *f* is the function given in (3), with $s\in[1,2)$, then $\dim_B \Gamma(f)=s$.

Proof. 1. We prove that $\overline{\dim}_B\Gamma(f)=s$.

The first part of the proof follows that of the theorem 1.

Let us consider $0<\beta<1$, small enough and $k\in\mathbf{N}^*$ such that:

$$\lambda_{k+1}^{-1}\le\beta<\lambda_k^{-1} \tag{4}$$

Then

$$\left|f(x+\beta)-f(x)\right|=\left|\sum_{i=1}^{\infty}\lambda_i^{s-2}\left\{g\left[\lambda_i(x+\beta)\right]-g(\lambda_i x)\right\}\right|\le$$

$$\le\sum_{i=1}^{k}\lambda_i^{s-2}\left|g\left[\lambda_i(x+\beta)\right]-g(\lambda_i x)\right|+\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\left|g\left[\lambda_i(x+\beta)\right]-g(\lambda_i x)\right|\le$$

$$\le\sum_{i=1}^{k}\lambda_i^{s-2}\left|g\left[\lambda_i(x+\beta)\right]-g(\lambda_i x)\right|+2\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\Rightarrow$$

$$\left|f(x+\beta)-f(x)\right|=2\left[\beta\sum_{i=1}^{k}\lambda_i^{s-1}+\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\right].$$

Using the Hadamard condition it can be deduced that

$$2\beta\sum_{i=1}^{k}\lambda_i^{s-1}<2C_1\beta\lambda_k^{s-1},$$

$$\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\le\sum_{j=0}^{\infty}(\varepsilon^j\lambda_{k+1})^{s-2}=\lambda_{k+1}^{s-2}\sum_{j=0}^{\infty}\varepsilon^{j(s-2)}<C_2\lambda_{k+1}^{s-2},$$

where C_1 and C_2 are constant that don’t depend on ε and *s*. Thus,

$$\left|f(x+\beta)-f(x)\right|=2\left(\beta C_1\lambda_k^{s-1}+C_2\lambda_{k+1}^{s-2}\right). \tag{5}$$

From (4) and (5) we obtain

$$\left|f(x+\beta)-f(x)\right|\le 2\beta C_1(\beta^{-1})^{s-1}+2C_2\beta^{2-s}=2(C_1+C_2)\beta^{2-s}. \tag{6}$$

From lemma 1 and (6), we deduce that

$$N_\beta \le 2m+\beta^{-1}\sum_{j=0}^{m-1}R_f\left[j\beta,(j+1)\beta\right]\le 2\beta^{-1}+\beta^{-1}\cdot 2\beta^{-1}\left(C_1+C_2\right)\beta^{2-s}\Leftrightarrow$$

$$N_\beta \le 2\beta^{-1}+2\left(C_1+C_2\right)\beta^{-s}.$$

Since $\beta\in(0,1)$ and $s\in[1,2)$, then $\beta^{-1}<\beta^{-s}$ and from the previous relation it results that

$$N_\beta \le C\beta^{-s}, \text{ where } C=2\left(1+C_1+C_2\right).$$

Therefore

$$\overline{\dim}_B\Gamma(f)=\overline{\lim_{\beta\to 0}}\frac{\log N_\beta}{-\log\beta}\le\lim_{\beta\to 0}\frac{\log C-s\log\beta}{-\log\beta}=s,$$

so, $\overline{\dim}_B\Gamma(f)\le s.$

2. We prove that $\underline{\dim}_B\Gamma(f)\ge s.$

If we consider $\beta\in(0,1)$ and $k\in\mathbf{N}^*$ such that

$$\lambda_{k+1}^{-1}\le\left(\varepsilon\lambda_k\right)^{-1}\le\beta<\lambda_k^{-1}. \tag{7}$$

and $j\in\mathbf{N}^*$, then

$$\left|f(j\beta)-f((j-1)\beta)\right|=\left|\sum_{i=1}^{\infty}\lambda_i^{s-2}\left\{g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right\}\right|\ge$$

$$\ge\left|\sum_{i=1}^{k}\lambda_i^{s-2}\left\{g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right\}\right|-\left|\sum_{i=k+1}^{\infty}\lambda_i^{s-2}\left\{g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right\}\right|\Rightarrow$$

$$\left|f(j\beta)-f((j-1)\beta)\right|=\left|\sum_{i=1}^{k}\lambda_i^{s-2}\left\{g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right\}\right|-C_2\lambda_{k+1}^{s-2}. \tag{8}$$

We shall estimate the modulus from the formula (7), using lemma 2.

Let $p\in(0,1)$ be any number and $q=\frac{p}{p-1}$. Then

$$\left|\sum_{i=1}^{k}\lambda_i^{s-2}\left\{g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right\}\right|\ge\left[\sum_{i=1}^{k}\left(\lambda_i^{s-2}\right)^p\right]^{\frac{1}{p}}\left[\sum_{i=1}^{k}\left|g\left(\lambda_i j\beta\right)-g\left(\lambda_i(j-1)\beta\right)\right|^q\right]^{\frac{1}{q}}. \tag{9}$$

Using the Hadamard conditions and $s\in[1,2)$, we obtain

$$\lambda_k > \varepsilon\lambda_{k-1} > \ldots > \varepsilon^{k-1}\lambda_1 \Rightarrow \lambda_k^{s-2} < \varepsilon^{s-2}\lambda_{k-1}^{s-2} < \ldots < \varepsilon^{(s-2)(k-1)}\lambda_1^{s-2} \Rightarrow$$

$$\sum_{i=1}^{k} \lambda_i^{p(s-2)} > \varepsilon^{p(2-s)} \frac{\varepsilon^{p(2-s)k} - 1}{\varepsilon^{p(2-s)} - 1} \lambda_k^{p(s-2)} > \lambda_k^{p(s-2)} \Rightarrow$$

$$\left[\sum_{i=1}^{k} \lambda_i^{p(s-2)}\right]^{\frac{1}{p}} > \lambda_k^{s-2} > \beta^{2-s}. \tag{10}$$

A point x is called an exceptional point for a function g if the derivative $g'(x)$ doesn't exist.

For the exceptional points $\lambda_i j\beta$, $\lambda_i (j-1)\beta$,

$$\left|g(\lambda_i j\beta) - g(\lambda_i (j-1)\beta)\right| = 2\beta\lambda_i .$$

For the non-exceptional point x_0, for every $\varepsilon_0 > 0$,

$$\left|\frac{g(x_0 + h) - g(x_0)}{h} - g'(x_0)\right| < \varepsilon_0 \text{, when } h \to 0 .$$

Then

$$g'(x_0)h - \varepsilon_0 h < g(x_0 + h) - g(x_0) < g'(x_0)h + \varepsilon_0 h, h \to 0 .$$

If $g'(x_0) = 2$, then

$$0 < (2 - \varepsilon_0)h < g(x_0 + h) - g(x_0) < (2 + \varepsilon_0)h \Rightarrow \left|g(x_0 + h) - g(x_0)\right| > (2 - \varepsilon_0)h .$$

If $g'(x_0) = -2$, then

$$-(2 + \varepsilon_0)h < g(x_0 + h) - g(x_0) < (-2 + \varepsilon_0)h < 0 \Rightarrow \left|g(x_0 + h) - g(x_0)\right| > (2 - \varepsilon_0)h.$$

Therefore,

$$\left|g(x_0 + h) - g(x_0)\right| > (2 - \varepsilon_0)h,\ h \to 0 .$$

Particularly, for the non-exceptional points $\lambda_i (j-1)\beta$,

$$\left|g(\lambda_i j\beta) - g(\lambda_i (j-1)\beta)\right| > (2 - \varepsilon_0)\lambda_i\beta ,$$

for $\lambda_i\beta$ small enough. So, for $\varepsilon_0 \to 0$,

$$\left[\sum_{i=1}^{k} \left|g(\lambda_i j\beta) - g(\lambda_i (j-1)\beta)\right|^q\right]^{\frac{1}{q}} \ge C_4\beta\left[\sum_{i=1}^{k} \lambda_i^q\right]^{\frac{1}{q}} .$$

Since $q < 0$,

$$\sum_{i=1}^{k} \lambda_i^q > \lambda_k^q \frac{1 - \varepsilon^{kq}}{1 - \varepsilon^q} > \lambda_k^q$$

and using (7), it can be seen that

$$\left[\sum_{i=1}^{k}\left|g(\lambda_i j\beta)-g(\lambda_i (j-1)\beta)\right|^q\right]^{\frac{1}{q}} \geq C_4\beta\lambda_k \geq C_5 . \tag{11}$$

Thus, the relations (8) - (11) give:

$$\left|f(j\beta)-f((j-1)\beta)\right| \geq C\beta^{2-s}, C>0 .$$

Now, from lemma 1,

$$N_\beta \geq \beta^{-1}\sum_{j=1}^{m} R_f\left[(j-1)\beta, j\beta\right] \Leftrightarrow N_\beta \geq Cm\beta^{-1}\beta^{2-s} \geq C\beta^{1-s}\left(\beta^{-1}-1\right)=C\beta^{-s}(1-\beta) \Rightarrow$$

$$\underline{\lim}_{\beta\to 0}\frac{\log N_\beta}{-\log\beta} \geq \underline{\lim}_{\beta\to 0}\frac{\log[C(1-\beta)]}{-\log\beta}+s,$$

thus, $\underline{\dim}_B\Gamma(f)\geq s.$

From the parts 1 and 2, it results that $\dim_B \Gamma(f)=s.$

Corollary 1 If g is the function given in (2), $s\in[1,2)$ and

$$f(x)=\sum_{i=1}^{\infty}\left(\lambda^i\right)^{s-2} g\left(\lambda^i x\right), \lambda>1,$$

then $\dim_B \Gamma(f)=s.$

Corollary 2 If g is any periodic zig - zag function, $s\in[1,2)$ and

$$f(x)=\sum_{i=1}^{\infty}\left(\lambda^i\right)^{s-2} g\left(\lambda^i x\right), \lambda>1,$$

then $\dim_B \Gamma(f)=s.$

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